

PHY 711: ANALYTICAL DYNAMICS
Additional Practice Problems II

Problem 1

A particle of mass m starts from the north pole and moves with constant speed v along a longitude to reach the south pole of Earth. We can consider Earth to be a sphere of mass M and radius R which rotates freely in space. Earth's initial angular velocity when the particle starts its trip is ω . Show that when the particle reaches the south pole, the rotation of Earth will have been retarded by an angle

$$\alpha = \frac{\pi\omega R}{v} \left[1 - \left(\frac{2M}{2M + 5m} \right)^{\frac{1}{2}} \right]$$

Solution

We use the usual spherical coordinates with θ as the latitude varying from zero to π and φ as the azimuthal angle varying from zero to 2π . The motion along the θ -direction is given to be at a constant speed v , so

$$R\dot{\theta}(t) = vt \implies \theta(t) = \frac{v}{R}t$$

The Lagrangian for the motion along φ is

$$L = \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}mR^2 \sin^2 \theta \dot{\varphi}^2$$

This gives the equation of motion

$$\frac{\partial}{\partial t} [(I + mR^2 \sin^2 \theta)\dot{\varphi}] = 0 \implies (I + mR^2 \sin^2 \theta)\dot{\varphi} = l$$

Since l is a constant of motion, we can evaluate it at $t = 0$, i.e., at $\theta = 0$, when $\dot{\varphi} = \omega$. This gives $l = I\omega$. The angle turned over by Earth as θ changes from zero to π is thus

$$\begin{aligned} \varphi &= \int_0^{\pi R/v} dt \dot{\varphi} = \frac{\omega R}{v} \int_0^{\pi} d\theta \frac{1}{1 + (mR^2/I) \sin^2 \theta} \\ &= 2 \frac{\omega R}{v} \int_0^{\pi/2} d\theta \frac{1}{1 + (mR^2/I) \sin^2 \theta} \end{aligned}$$

Let $b = mR^2/I = 5m/(2M)$, using $I = (2/5)MR^2$. The integral can be evaluated as follows. Since $d(\tan \theta) = d\theta \sec^2 \theta = d\theta / \cos^2 \theta$, we can write

$$\int_0^{\pi/2} d\theta \frac{1}{1 + (mR^2/I) \sin^2 \theta} = \int_0^{\pi/2} d\theta \frac{1}{\cos^2 \theta [1 + (1+b) \tan^2 \theta]}$$

$$\begin{aligned}
&= \int d(\tan \theta) \frac{1}{[1 + (1 + b) \tan^2 \theta]} \\
&= \int_0^\infty du \frac{1}{[1 + (1 + b)u^2]} = \frac{\pi}{2} \frac{1}{\sqrt{1 + b}}
\end{aligned}$$

This gives

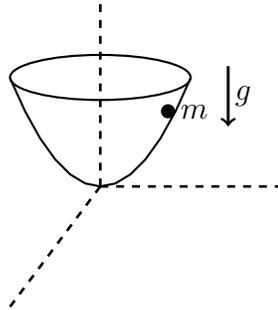
$$\varphi = \frac{\pi\omega R}{v} \sqrt{\frac{2M}{2M + 5m}}$$

In the absence of m , Earth would have turned by $\pi\omega R/v$ (obtained by setting $m = 0$ above). The amount of retardation is thus

$$\alpha = \frac{\pi\omega R}{v} - \frac{\pi\omega R}{v} \sqrt{\frac{2M}{2M + 5m}} = \frac{\pi\omega R}{v} \left[1 - \left(\frac{2M}{2M + 5m} \right)^{\frac{1}{2}} \right]$$

Problem 2

A small ball of radius R can roll on the inner surface of a bowl in the shape of a paraboloid placed on the ground as shown. The paraboloid is described by $z = \frac{1}{2}\lambda r^2$, where $r^2 = x^2 + y^2$, and λ is a positive constant. Write down the Lagrangian describing the motion of the ball. (*Hint:* Use cylindrical coordinates for which $ds^2 = dz^2 + dr^2 + r^2 d\varphi^2$. Moment of inertia for a sphere = $(2/5)MR^2$)



Problem 2

Solution

The kinetic energy will have contributions from the motion of the center of mass and from the rotational rolling motion. Let χ denote the angle the ball rolls through in a certain amount of time. Then, since there is no slipping, we have the relation

$$R\chi = \text{distance moved in the same amount of time} \equiv s$$

Thus $\dot{\chi} = \dot{s}/R$. The total kinetic energy is given by

$$T = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}I\dot{\chi}^2 = \frac{1}{2}\left(M + \frac{2}{5}M\right)\dot{s}^2 = \frac{7}{10}M(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)$$

The potential energy is $V = Mgz$. We also have $z = \frac{1}{2}\lambda r^2$, so that $\dot{z} = \lambda r \dot{r}$. Thus

$$L = \frac{7}{10}M((1 + \lambda^2 r^2)\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{1}{2}Mg\lambda r^2$$

I asked for the Lagrangian, so this is the answer. But it is trivial to work out the equations of motion as well. We find

$$\begin{aligned}\frac{\partial L}{\partial \dot{r}} &= \frac{7}{5}M(1 + \lambda^2 r^2)\dot{r}, & \frac{\partial L}{\partial \dot{\varphi}} &= \frac{7}{5}Mr^2\dot{\varphi} \\ \frac{\partial L}{\partial r} &= \frac{7}{5}(Mr\dot{\varphi}^2 + M\lambda^2 r\dot{r}^2) - Mg\lambda r, & \frac{\partial L}{\partial \varphi} &= 0\end{aligned}$$

The equations of motion are then

$$\begin{aligned}\frac{7}{5}[(1 + \lambda^2 r^2)\ddot{r} + \lambda^2 r \dot{r}^2 - r\dot{\varphi}^2] + g\lambda r &= 0 \\ \frac{d}{dt}\left[\frac{7}{5}Mr^2\dot{\varphi}\right] &= 0\end{aligned}$$

Problem 3

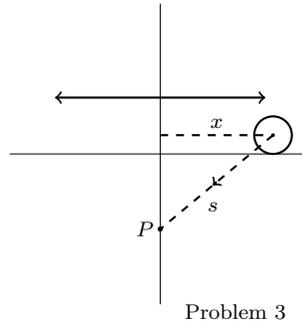
A disk of mass M and radius R can roll on a horizontal plane along the x -axis. (Ignore motion along the other directions.) It is attracted to a fixed point P with a force $F = -ks$ where s is the distance of the center of the disk from P (See figure). Obtain the Lagrangian and the equations of motion. The disk can oscillate (in a rolling motion) as shown. Obtain the frequency of this oscillation. (Moment of inertia of a disk for rotations around its axis is $\frac{1}{2}MR^2$. (*Hint: Obtain the potential energy corresponding to F and then use the geometry to relate s to x . To obtain the potential energy, it is easier to consider the force split into its components along the horizontal and vertical axes.*)

Solution

The given force has magnitude $-ks$ and the direction is towards the point P as in figure. The components of F may be written as

$$F_1 = -kx, \quad F_2 = -ky$$

Using $\vec{F} = -\nabla V$, we see that the potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. The motion in the y -direction is immaterial since the motion is restricted to the x -direction. So we do not really



need the term $\frac{1}{2}ky^2$. The gravitational potential energy Mgy is also irrelevant. The kinetic energy has two contributions, from the translational motion of the center of mass and the rotational motion. Further we have the relation $R\varphi = x$, connecting the distance moved with the angle turned through, since there is no slipping. Thus

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\varphi}^2 - \frac{1}{2}kx^2 + \text{irrelevant terms} = \frac{3}{4}M\dot{x}^2 - \frac{1}{2}kx^2 + \dots$$

The equation of motion is

$$\frac{3}{2}M\ddot{x} + kx = 0 \implies \ddot{x} + \omega^2 x = 0$$

where $\omega = \sqrt{2k/3M}$. This equation describes simple harmonic oscillation with frequency ω and the solution is

$$x(t) = A \cos \omega t + B \sin \omega t$$

Problem 4

The motion of a class of asteroids shepherded by the planet Jupiter can be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}m \left[(\dot{Q}_1 - \omega Q_2)^2 + (\dot{Q}_2 + \omega Q_1)^2 \right] + \frac{\mu_1}{\sqrt{(Q_1 - \alpha_2 R)^2 + Q_2^2}} + \frac{\mu_2}{\sqrt{(Q_1 + \alpha_1 R)^2 + Q_2^2}}$$

where Q_1, Q_2 are the position coordinates in the plane of the orbit and ω is the orbital angular velocity of Jupiter, taken as constant. $\mu_1, \mu_2, \alpha_1, \alpha_2, R$ are all constants. Obtain the Hamiltonian and the canonical equations of motion. (*There are two groups of asteroids, called the Trojans and the Greeks, one of which is 60° ahead of Jupiter and the other is 60° behind Jupiter in its orbit which are kept in the same relative position by the combined gravitational force of the Sun and Jupiter and the centrifugal effect of their revolution around the Sun. This is a classic problem of celestial mechanics. The given Lagrangian describes the dynamics of these asteroids.*)

Solution

There are two generalized coordinates Q_1 and Q_2 . We then find

$$P_1 = \frac{\partial L}{\partial \dot{Q}_1} = m(\dot{Q}_1 - \omega Q_2), \quad P_2 = \frac{\partial L}{\partial \dot{Q}_2} = m(\dot{Q}_2 + \omega Q_1)$$

Solving for the velocities in terms of the canonical momenta, we get

$$\dot{Q}_1 = \frac{P_1}{m} + \omega Q_2, \quad \dot{Q}_2 = \frac{P_2}{m} - \omega Q_1$$

Thus the Hamiltonian is given by

$$\begin{aligned} H &= P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - L \\ &= P_1 \left[\frac{P_1}{m} + \omega Q_2 \right] + P_2 \left[\frac{P_2}{m} - \omega Q_1 \right] - \frac{m}{2} [(P_1/m)^2 + (P_2/m)^2] \\ &\quad - \frac{\mu_1}{\sqrt{(Q_1 - \alpha_2 R)^2 + Q_2^2}} - \frac{\mu_2}{\sqrt{(Q_1 + \alpha_1 R)^2 + Q_2^2}} \\ &= \frac{P_1^2 + P_2^2}{2m} + \omega(P_1 Q_2 - P_2 Q_1) - \frac{\mu_1}{\sqrt{(Q_1 - \alpha_2 R)^2 + Q_2^2}} - \frac{\mu_2}{\sqrt{(Q_1 + \alpha_1 R)^2 + Q_2^2}} \end{aligned}$$

The canonical equations of motion are:

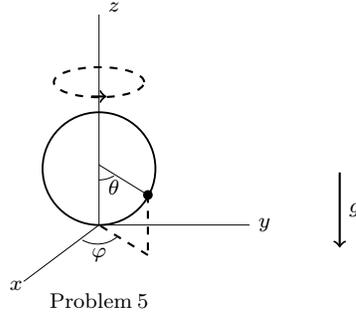
$$\begin{aligned} \dot{Q}_1 &= \frac{\partial H}{\partial P_1} = \frac{P_1}{m} + \omega Q_2 \\ \dot{Q}_2 &= \frac{\partial H}{\partial P_2} = \frac{P_2}{m} - \omega Q_1 \\ \dot{P}_1 &= -\frac{\partial H}{\partial Q_1} = \omega P_2 - \frac{\mu_1(Q_1 - \alpha_2 R)}{[(Q_1 - \alpha_2 R)^2 + Q_2^2]^{\frac{3}{2}}} - \frac{\mu_2(Q_1 + \alpha_1 R)}{[(Q_1 + \alpha_1 R)^2 + Q_2^2]^{\frac{3}{2}}} \\ \dot{P}_2 &= -\frac{\partial H}{\partial Q_2} = -\omega P_1 - \frac{\mu_1 Q_2}{[(Q_1 - \alpha_2 R)^2 + Q_2^2]^{\frac{3}{2}}} - \frac{\mu_2 Q_2}{[(Q_1 + \alpha_1 R)^2 + Q_2^2]^{\frac{3}{2}}} \end{aligned}$$

Problem 5

Consider a hoop of radius R placed vertically on a horizontal flat frictionless surface. It can rotate around the vertical axis. A bead of mass m is free to slide on the hoop, see figure.

- Obtain the Lagrangian of the system for the bead and the hoop.
- Obtain the equations of motion.
- Identify the canonical momenta and work out the Hamiltonian.
- Obtain the canonical equations of motion.

(The rotation of the hoop is not externally controlled, it is part of the dynamics. The moment of inertia for a hoop of mass M for rotation around one of its diameters is $I = \frac{1}{2}MR^2$.)



Solution

The coordinates of the bead in a general position will be given by

$$x = R \sin \theta \cos \varphi, \quad y = R \sin \theta \sin \varphi, \quad z = R - R \cos \theta$$

The kinetic energy of the bead is thus

$$T_{\text{bead}} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}R^2 [\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2]$$

The kinetic energy for the hoop itself is

$$T_{\text{hoop}} = \frac{1}{2}I\dot{\varphi}^2 = \frac{1}{4}MR^2\dot{\varphi}^2$$

The total kinetic energy is thus

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}R^2[(M/2) + m \sin^2 \theta]\dot{\varphi}^2$$

The potential energy is

$$V = mgz = mgR(1 - \cos \theta)$$

The Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}R^2[(M/2) + m \sin^2 \theta]\dot{\varphi}^2 + mgR \cos \theta + \text{constant}$$

We then get

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= p_{\theta} = mR^2\dot{\theta} \\ \frac{\partial L}{\partial \dot{\varphi}} &= p_{\varphi} = \left(\frac{1}{2}M + m \sin^2 \theta\right)R^2\dot{\varphi} \\ \frac{\partial L}{\partial \theta} &= mR^2 \sin \theta \cos \theta \dot{\varphi}^2 - mgR \sin \theta \\ \frac{\partial L}{\partial \varphi} &= 0 \end{aligned}$$

The Lagrangian equations of motion are:

$$mR^2\ddot{\theta} = mR^2 \sin \theta \left(\cos \theta \dot{\varphi}^2 - \frac{g}{R} \right)$$

$$\frac{d}{dt} \left[\left(\frac{1}{2}M + m \sin^2 \theta \right) \dot{\varphi} \right] = 0$$

The canonical momenta are p_θ and p_φ as given above. The Hamiltonian is then obtained as

$$H = p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - L$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_\varphi^2}{2\left(\frac{1}{2}M + m \sin^2 \theta\right)R^2} - mgR \cos \theta + \text{constant}$$

The canonical equations of motion are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$

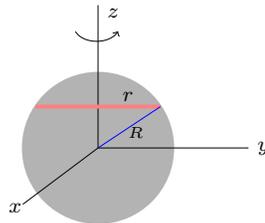
$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{\left(\frac{1}{2}M + m \sin^2 \theta\right)R^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2}{R^2\left(\frac{1}{2}M + m \sin^2 \theta\right)^2} m \sin \theta \cos \theta - mgR \sin \theta$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

Problem 6

Consider a uniform circular disc of small thickness, mass M and radius R . Calculate its moment of inertia for rotations around one of the diameters; i.e., calculate $I_{33} = I_{zz}$ for the disc shown in figure. (This is like a coin spinning on its edge.) (You may encounter integrals like $\int d\theta \sin^4 \theta$ or $\int d\theta \cos^4 \theta$. You can simplify them using trigonometric identities expressing $\sin^2 \theta$ and $\cos^2 \theta$ in terms of $\cos 2\theta$.)



Problem 6

Solution

Consider a thin slice of the disk, as indicated in red. The thickness, taken along the x -direction, is δ . The value of y changes from $-r$ to r , where $r = \sqrt{R^2 - z^2}$, R being the

radius of the disk. We then get

$$\begin{aligned}
 I_{33} &= \int dV \rho (x^2 + y^2) = \rho \delta \int dz dy y^2 + \mathcal{O}(\delta^3) \\
 &= \rho \delta \int_{-R}^R dz \frac{y^3}{3} \Big|_{-r}^r = \frac{2}{3} \rho \delta \int_{-R}^R dz (R^2 - z^2)^{\frac{3}{2}} \\
 &= \frac{2}{3} \rho \delta R^4 \int_0^\pi d\theta \sin \theta \sin^3 \theta
 \end{aligned}$$

In the last line, we used the substitution $z = R \cos \theta$. We can now write

$$\sin^4 \theta = \left(\frac{1 - \cos 2\theta}{2} \right)^2 = \frac{1}{4} \left[1 - 2 \cos 2\theta + \frac{1}{2} (1 + \cos 4\theta) \right]$$

The integrals of $\cos 2\theta$ and $\cos 4\theta$ will vanish at the limits of integration, so we get

$$I_{33} = \frac{2}{3} \rho \delta R^4 \left(\frac{1}{4} + \frac{1}{8} \right) \pi = \frac{1}{4} (\pi R^2 \delta \rho) R^2 = \frac{1}{4} M R^2$$

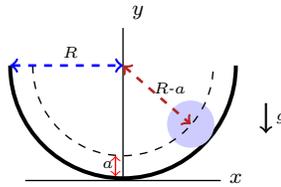
Problem 7

Consider a rail bent into a semicircular shape, of radius R , and placed on the ground as shown in figure. A disk of mass M and radius a can roll without slipping on the inside of this rail. (The motion is effectively one dimensional.)

- Obtain the Lagrangian for the motion of the disk. (Moment of inertia of disk is $\frac{1}{2} M a^2$.)
- Obtain the Hamiltonian and the canonical equations of motion.

(*Caution:* The distance traced out by the center of mass of the disk is different from the distance it rolls because a is not negligible.)

- Determine the frequency for small oscillations around the equilibrium point if $a = R/3$.



Problem 7

Solution

- Taking the line from the center of the circular rail to the center of the disk to be at an angle φ from the vertical line, we find the coordinates of the center of mass as

$$(X, Y) = (R - a) \sin \varphi, -(R - a) \cos \varphi$$

Let the angle turned by the wheel as it rolls be designated as α . The distance covered in the rolling motion, traced out on the inside of the rail, is $s = a\alpha$. This should equal the distance along the rail, i.e., $R\varphi$ if there is no slipping. Thus we have $a\alpha = R\varphi$. The kinetic energy is thus given by

$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I\dot{\alpha}^2 = \frac{1}{2}M(R-a)^2\dot{\varphi}^2 + \frac{1}{4}Ma^2\left(\frac{R\dot{\varphi}}{a}\right)^2 = \frac{1}{2}\mu\dot{\varphi}^2$$

$$\mu = M\left((R-a)^2 + \frac{R^2}{2}\right)$$

The potential energy is $V = MgY$, so that the Lagrangian is

$$L = \frac{1}{2}\mu\dot{\varphi}^2 + Mg(R-a)\cos\varphi$$

b) The canonical momentum for φ is

$$p = \frac{\partial L}{\partial \dot{\varphi}} = \mu\dot{\varphi} \implies \dot{\varphi} = p/\mu$$

Thus the Hamiltonian is

$$H = p\dot{\varphi} - L = \frac{p^2}{2\mu} - Mg(R-a)\cos\varphi$$

The canonical equations of motion are given by

$$\dot{\varphi} = \frac{\partial H}{\partial p} = p/\mu, \quad \dot{p} = -\frac{\partial H}{\partial \varphi} = -Mg(R-a)\sin\varphi$$

These may be combined as

$$\ddot{\varphi} = -\frac{Mg(R-a)}{\mu}\sin\varphi$$

c) For small amplitudes oscillations, this reduces to

$$\ddot{\varphi} \approx -\omega^2\varphi, \quad \omega^2 = \frac{Mg(R-a)}{\mu}$$

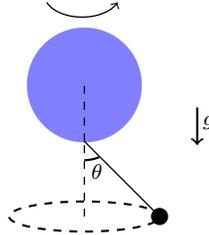
For $a = R/3$, $\mu = (17/18)MR^2$, $R-a = (2/3)R$, so that

$$\omega = \sqrt{\frac{12g}{17R}}$$

Problem 8

A solid spherical ball of mass M and radius R floats at some height above the ground. It is kept at a fixed height by some means not relevant to the problem. (You may think of it

as a spherical approximation to a hot air balloon.) From the bottom of the ball is hung a pendulum with a bob of mass m , using a thin rigid rod of negligible mass and fixed length l . The mass m can move in θ and in the azimuthal angle. The ball itself can rotate around a vertical axis. (We will ignore other types of motion for the ball.) Obtain the Lagrangian and the equations of motion for the system. (Keep in mind that there can be independent rotational motion for the ball and the bob of the pendulum.)



Problem 8

Solution

We take the center of the sphere as the origin of the coordinate system and let α denote the angle for rotation of the sphere. The point of suspension has coordinates $(0, 0, -R)$. The position of the bob is then given by $(0, 0, -R) + (l \sin \theta \cos(\alpha + \varphi), l \sin \theta \sin(\alpha + \varphi), -l \cos \theta)$, where φ denotes the possible additional azimuthal angle for the pendulum. The corresponding velocity is

$$\begin{aligned} \dot{x}_1 &= l \dot{\theta} \cos \theta \cos(\alpha + \varphi) - l(\dot{\alpha} + \dot{\varphi}) \sin \theta \sin(\alpha + \varphi) \\ \dot{x}_2 &= l \dot{\theta} \cos \theta \sin(\alpha + \varphi) + l(\dot{\alpha} + \dot{\varphi}) \sin \theta \cos(\alpha + \varphi) \\ \dot{x}_3 &= l \dot{\theta} \sin \theta \end{aligned}$$

The kinetic energy is thus given by

$$\begin{aligned} T &= \frac{1}{2} I \dot{\alpha}^2 + \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ &= \frac{MR^2}{5} \dot{\alpha}^2 + \frac{m}{2} \left[l^2 \dot{\theta}^2 + (\dot{\alpha} + \dot{\varphi})^2 l^2 \sin^2 \theta \right] \end{aligned}$$

where we use the fact, for just rotational motion, the kinetic energy of the ball $\frac{1}{2} I \dot{\alpha}^2$, with $I = (2/5)MR^2$. The potential energy is $V = mgz = -mg(R + l \cos \theta)$. Thus

$$L = \frac{MR^2}{5} \dot{\alpha}^2 + \frac{m}{2} \left[l^2 \dot{\theta}^2 + (\dot{\alpha} + \dot{\varphi})^2 l^2 \sin^2 \theta \right] + mgl \cos \theta + \text{constant}$$

The required derivatives for the equations of motion are

$$\frac{\partial L}{\partial \dot{\alpha}} = (I + ml^2 \sin^2 \theta) \dot{\alpha} + ml^2 \sin^2 \theta \dot{\varphi}, \quad \frac{\partial L}{\partial \alpha} = 0$$

$$\begin{aligned}\frac{\partial L}{\partial \dot{\varphi}} &= ml^2 \sin^2 \theta (\dot{\alpha} + \dot{\varphi}), & \frac{\partial L}{\partial \varphi} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2 \dot{\theta}, & \frac{\partial L}{\partial \theta} &= ml^2 (\dot{\alpha} + \dot{\varphi})^2 \sin \theta \cos \theta - mgl \sin \theta\end{aligned}$$

The equations of motion are thus

$$\begin{aligned}\frac{d}{dt} [(I + ml^2 \sin^2 \theta) \dot{\alpha} + ml^2 \sin^2 \theta \dot{\varphi}] &= 0 \\ \frac{d}{dt} [ml^2 \sin^2 \theta (\dot{\alpha} + \dot{\varphi})] &= 0 \\ ml^2 \ddot{\theta} &= ml^2 (\dot{\alpha} + \dot{\varphi})^2 \sin \theta \cos \theta - mgl \sin \theta\end{aligned}$$

Problem 9

A rigid body with principal moments of inertia I_1 and $I_2 = I_3$ moves under the action of a torque

$$\tau_1 = 0, \quad \tau_2 = A \cos \omega t, \quad \tau_3 = A \sin \omega t$$

Find the angular velocities $\Omega_1, \Omega_2, \Omega_3$ as functions of time, for arbitrary initial conditions.

Solution

Euler's equations for a rigid body, with $I_2 = I_3$ and the given torque, become

$$\begin{aligned}I_1 \dot{\Omega}_1 &= 0 \\ \dot{\Omega}_2 + \alpha \Omega_3 &= \frac{A}{I_3} \cos \omega t \\ \dot{\Omega}_3 - \alpha \Omega_2 &= \frac{A}{I_3} \sin \omega t\end{aligned}$$

where $\alpha = (I_1 - I_3)\Omega_1/I_3$. The first equation tells us that Ω_1 is a constant of motion. Thus α is a constant as well. We add i times the last equation to the second one to get

$$\dot{Q} - i\alpha Q = \frac{A}{I_3} e^{i\omega t}$$

where $Q = \Omega_2 + i\Omega_3$ and we used $e^{i\omega t} = \cos \omega t + i \sin \omega t$. Writing $Q = e^{i\alpha t} F$, we find

$$\dot{F} = \frac{A}{I_3} e^{i(\omega-\alpha)t} \implies F = \frac{A}{I_3 i(\omega-\alpha)} e^{i(\omega-\alpha)t} + C$$

Using $Q = e^{i\alpha t} F$ and writing $C = |C|e^{i\varphi_0}$, we can separate the real and imaginary parts to get

$$\Omega_2 = \frac{A}{I_3 (\omega - \alpha)} \sin \omega t + |C| \cos(\alpha t + \varphi_0)$$

$$\begin{aligned}\Omega_3 &= -\frac{A}{I_3(\omega - \alpha)} \cos \omega t + |C| \sin(\alpha t + \varphi_0) \\ \Omega_1 &= \text{constant}\end{aligned}$$

This gives the complete solution.

Problem 10

Show that the following change of variables is a canonical transformation.

$$Q = \log \left(\frac{1}{q} \sin p \right), \quad P = q \cot p$$

Solution

The easiest way to solve this is to demonstrate that the Poisson brackets are unchanged. In terms of q, p , the PB is defined by

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$$

Now we take f, g to be given in terms of Q and P , and write the above definition of the PB in terms of derivatives with respect to Q, P .

$$\begin{aligned}\frac{\partial f}{\partial q} &= \frac{\partial f}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P} \frac{\partial P}{\partial q} = \frac{\partial f}{\partial Q} (-1/q) + \frac{\partial f}{\partial P} \cot p \\ \frac{\partial g}{\partial p} &= \frac{\partial g}{\partial Q} \frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P} \frac{\partial P}{\partial p} = \frac{\partial g}{\partial Q} \cot p + \frac{\partial g}{\partial P} (-q \csc^2 p)\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} &= \left(\frac{\partial f}{\partial Q} (-1/q) + \frac{\partial f}{\partial P} \cot p \right) \left(\frac{\partial g}{\partial Q} \cot p + \frac{\partial g}{\partial P} (-q \csc^2 p) \right) - (f \leftrightarrow g) \\ &= \frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} (\csc^2 p) + \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \cot^2 p - (f \leftrightarrow g) \\ &= \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) (\csc^2 p - \cot^2 p) \\ &= \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) = \{f, g\}_{Q,P}\end{aligned}$$

The PB calculated in terms of q, p , agrees with the PB defined in terms of Q, P . Hence the given change of variables is canonical.